AD-A031 410

WVT-TR-76038

SOLUTIONS TO INITIAL VALUE PROBLEMS USING FINITE ELEMENTS - UNCONSTRAINED VARIATIONAL FORMULATIONS

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September 1976

U.S. ARMY ARMAMENT COMMAND

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WATERVLIET ARSENAL
WATERVLIET, N.Y. 12189

TECHNICAL REPORT

AMCMS No. 611101.11.84400

DA Project No. 1T161101A91A

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER WVT-TR-76038	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SOLUTIONS TO INITIAL VALUE PROBLEMS USING FINITE ELEMENTS - UNCONSTRAINED VARIATIONAL FORMULATIONS		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J. J. Wu		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Benet Weapons Laboratory Watervliet Arsenal, Watervliet, N.Y. 12189 SARWV-RT-TP		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS AMCMS No. 611101.11.84400 DA Proj. No. 1T161101A91A
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Armament Command Rock Island, Illinois 61201		12. REPORT DATE September 1976 13. NUMBER OF PAGES 27
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Finite Element
Initial Value Problem
Variational Principles
Vibration
Heat Conduction
Wave Propagation
Method of Approximation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper presents a variational formulation which treats initial value problems and boundary problems in a unified manner. The basic ingredients of this theory are (1) adjoint variable and (2) unconstrained variations. It is an extension of the finite element-unconstrained variational formulation used previously in solving several nonconservative stability problems. The technique which makes this extension possible is described. This formulation thus enables one to adapt such numerical technique as the finite element

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method, which has had great success and popularity for solution of boundary value problems, for solutions of initial value problems as well. These formulations are given here for a forced vibration problem, a heat (mass) transfer problem and a wave propagation problem. Numerical calculations in conjunction with finite elements for two specific examples are obtained and compared with known exact solutions.

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1. INTRODUCTION. In its application to the solutions of engineering problems, the finite element discretization has been implemented almost exclusively to the spatial dimensions. For dynamic or time-dependent problems whose solutions as functions of time are of interest, a step-by-step procedure of finite difference, i.e., the quasi-static approach is usually employed. The answer to the question why the time dimension has not been treated equally with the spatial variables in the finite element discretization must be related, in part at least, to the development of variational methods, since the finite element procedure can be viewed most readily as an extremizing sequence associated with a variational statement. While there are numerous variational principles for boundary value problems, few exist for initial value problems. Like many problems involving nonconservative forces, the difficulty appears to be that initial value problems are nonself-adjoint and thus they do not possess variational principles in the classical sense. In conjunction with problems involving nonconservative forces, certain constrained variational principles (sometimes called extended Hamilton's principles See, for example, ref. [1]) were used for finite element solution formulations [2, 3]. Shortly afterwards, using the combined notion of the Lagrange multipliers and the adjoint variable, some unconstrained variational statements were established and used as bases for finite element solutions [4, 5]. This approach has been shown to be more advantageous in terms of simplicity, versatility and the rate of convergence compared with the constrained variational approach [5, 6].

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1. Levinson, M., "Application of the Galerkin and Ritz Methods to Nonconservative Problems of Elastic Stability," Zeitschrift fur Angewandte Mathematic und Physik, Vol. 17, pp. 431-442 (1966).

2. Barsoum, R. S., "Finite Element Method Applied to the Problem of Stability of a Nonconservative System," International Journal for Numerical Methods in Engineering, Vol. 3, pp. 63-87 (1971).

3. Mote, C. D., "Nonconservative Stability by Finite Element," Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. EM3, pp. 645-656 (June 1971).

4. Wu, J. J., "Column Instability under Nonconservative Forces, with Internal and External Damping - Finite Element Using Adjoint Variational Principles," Development in Mechanics, Vol. 7, pp. 501-514 (1973).

5. Wu, J. J., "A Unified Finite Element Approach to Column Stability Problems," Development in Mechanics, Vol. 8, pp. 279-294 (1975).

6. Wu, J. J., "On the Numerical Convergence of Matrix Eigenvalue Problems Due to Constraint Conditions," Journal of Sound and Vibrations, Vol. 37, pp. 349-358 (1974).

Fried was first to treat the time-dimension identically with the space dimensions in using the finite elements [7]. His solution formulations, however, emanate from constrained variational principles. In contrast, this paper presents a generalization of the unconstrained variational approach to time-dependent problems.

At this point, the variational principles of integrals of convolution developed by Gurtin [8, 9] should be mentioned. The applications of these principles in conjunction with finite elements in the time-dimension [10, 11, 12, 13] have so far failed to show any advantage over the procedure described by Fried. In fact, all these analyses had to resort to either the Fried's or some other similar step-by-step procedure to complete the solutions in the time-dimension.

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7. Fried, I., "Finite Element Analysis of Time Dependent Phenomena," Journal of the American Institute of Aeronautics and Astronautics, Vol. 7, No. 6, pp. 1170-1172 (June 1970).

8. Gurtin, M. E., "Variational Principles for Linear Elastodynamics," Archive for Rational Mechanics and Analysis, Vol. 16, No. 1, pp. 36-50 (1964).

9. Gurtin, M. E., "Variational Principles for Linear Initial-Value Problems," Quarterly of Applied Mathematics, Vol. 22, pp. 252-256 (1964).

10. Wilson, E. L. and Nickel, R. E., "Application of the Finite Element Method to Heat Conduction Analysis," Nuclear Engineering and Design, Vol. 4, pp. 276-286 (1966).

11. Dunham, R. S., Nickel, R. E. and Strickler, D. C., "Integration Operators for Transient Structural Response," Computers and Structures, Vol. 2, pp. 1-15 (1972).

12. Ghaboussi, J. and Wilson, E. L., " Variational Formulation of Dynamics of Fluid-Saturated Porous Elastic Solids,: Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. EM4, pp. 947-963 (1972).

13. Atluri, S., "An Assumed Stress Hybrid Finite Element Model for Linear Elastodynamic Analysis," Journal of the American Institute of Aeronautics and Astronautics, Vol. 11, No. 7, pp. 1028-1031 (1973).

In this paper, the use of unconstrained variational principles - finite elements for usual boundary value problems is first illustrated and the advantages over the constrained formulations are pointed out. The unconstrained variational principles can always be constructed through the use of the Lagrange multipliers. The unconstrained variations are then shown to lead naturally to (nonself-) adjoint variational statements. Thus, nonconservative problems can be formulated easily using finite elements. The application to a control problem is given [14]. With the introduction of a cross-product term involving two-point boundary (initial) values, the unconstrained variational - finite element formulation is again easily extended to include time-dependent problems. This formulation is obviously simpler compared with those derived from Gurtin's variational principles because no convolutional integrals are needed. It is also easier to use and more versatile than the Fried's procedure due to the fact that no boundary or initial conditions are involved in the solution formulation and because of the nature of the Lagrange multipliers. As further examples of application, finite element matrix equations are derived for several transient problems including a force vibration, a heat transfer and a wave propagation problem. Detailed formulations and numerical results of two examples are given and comparisons with some known exact solutions are made.

2. <u>LAGRANGE MULTIPLIERS AND FINITE ELEMENT FORMULATIONS</u>. One of the advantages of the finite element method is its capability of solving large complicated problems in a routine manner. However, the same concepts used in a program for large systems may be understood using relatively simple problems.

Let us consider the stability of a Euler's column. The governing equations are as follows:

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14. Wu, J. J., "On the Stability of a Free-Free Beam under Axial Thrust Subjected to Directional Control," Journal of Sound and Vibration, Vol. 43, pp. 45-52 (1975). Also see a correction note on this paper, ibid, Vol. 44, p. 309 (1976).

D.E. $$E\,I\,u'''' + P\,u'' + \omega^2\,\rho A u = 0 \tag{1a}$$

B.C. $$u(0) = u'(0) = 0 \tag{1b),(1c}$$

$$u''(\ell) = 0 \tag{1d}$$

$$E\,I\,u'''(\ell) + P\,u'(\ell) = 0 \tag{1e}$$

A usual variational principle can be written

$$\delta J_1\ (u) = 0 \tag{2a}$$

where

$$J_1(u) = \frac{1}{2}\int_0^\ell [E\,I\,(u'')^2 - P(u')^2 + \omega^2\rho A u^2]\,dx \tag{2b}$$

To establish the equivalence between eqs. (1) and (2), one simply carries out the variation of J_1 in eq. (2a):

$$\delta J_1 = \int_0^\ell [E\,I\,u''\delta u'' - P\,u'\delta u' + \omega^2\,\rho A\,u\,\delta u]\,dx \tag{3a}$$

$$\begin{aligned} &= \int_0^\ell [E\,I\,u'''' + P\,u'' + \omega^2\,\rho A u]\,\delta u\,dx \\ &+ [E\,I\,u''\,\delta u' - (E\,I\,u''' + P\,u')\,\delta u]_{x\,=\,\ell} \\ &- [E\,I\,u''\,\delta u' - (E\,I\,u''' + P\,u')\,\delta u]_{x\,=\,0} \end{aligned} \tag{3b}$$

From eq. (3b) one observes that for the coordinate functions and their variations satisfying the boundary conditions in eqs. (1b - 1e), eq. (1a) implies eq. (2a) and vice versa. The finite element formulation for this problem begins with eq. (3a).

Let

$$u(x) = \underset{\sim}{a}^T(x)\,\underset{\sim}{U} \tag{4}$$

where $\underset{\sim}{a}(x)$ is the displacement-function vector and $\underset{\sim}{U}$, the generalized displacement vector. Upon the substitution of eq. (4) into eq. (3a), one immediately obtains

$$\delta\underset{\sim}{U}^T \left\{ \underset{\sim}{K}_1 + \omega^2\underset{\sim}{M} \right\} \underset{\sim}{U} = 0 \tag{5}$$

where

$$\underset{\sim}{K}_1 = \int_0^\ell [E\,I\,\underset{\sim}{a}''\,\underset{\sim}{a}''^T - P\underset{\sim}{a}'\,\underset{\sim}{a}'^T]\,dx \tag{6a}$$

$$\underset{\sim}{M} = \int_0^\ell \rho A\,\underset{\sim}{a}\,\underset{\sim}{a}^T\,dx \tag{6b}$$

Eq. (5) is not yet ready to be solved since neither $\tilde{U}$ nor $\delta\tilde{U}$ consists of independent elements due to the boundary conditions requirements placed on u(x).

Let us now consider a slightly different variational principle:

$$\delta J_2 = 0 \tag{7a}$$

with

$$J_2 = \frac{1}{2} \int_0^{\ell} [E\ I\ (u'')^2 - P\ (u')^2 + \omega^2\ \rho A u^2]\ dx$$
$$+ \frac{1}{2}\ \alpha_1\ [u(0)]^2 + \frac{1}{2}\ \alpha_2\ [u'(0)]^2 \tag{7b}$$

where α_1 and α_2 are the Lagrange multipliers.

Carrying out the variation of eqs. (7), we have

$$\delta J_2 = \int_0^{\ell} [E\ I\ u''\ \delta u'' - Pu'\ \delta u + \omega^2 \rho A u^2]\ dx$$
$$+ \alpha_1\ u(0)\ \delta u(0) + \alpha_2\ u'(0)\ \delta u'(0) \tag{8a}$$
$$= \int_0^{\ell} [E\ I\ u'''' + Pu'' + \omega^2\ \rho A u]\ \delta u\ dx$$
$$+ [E\ I\ u''\ \delta u' - (E\ I\ u''' + Pu')\ \delta u]_{x=\ell}$$
$$- [(E\ I\ u'' - \alpha_1 u' - (E\ I\ u''' + Pu' + \alpha_1 u)\ \delta u]_{x\ =\ 0} \tag{8b}$$

Eq. (8b) states that a necessary and sufficient condition for $\delta J_2 = 0$ is the problem defined by the following set of equations:

$$E\ I\ u'''' + Pu'' + \omega^2 \rho A u = 0 \tag{9a}$$
$$E\ I\ u''(0) - \alpha_2\ u'(0) = 0 \tag{9b}$$
$$E\ I\ u'''(0) + Pu'(0) + \alpha_1\ u(0) = 0 \tag{9c}$$
$$E\ I\ u''(\ell) = 0 \tag{9d}$$
$$E\ I\ u'''\ (\ell) + Pu'\ (\ell) = 0 \tag{9e}$$

provided that the variation δu is completely arbitrary, comparing eqs. (9) and (1), it is seen that eqs. (1) is a special case of (9) as α_1, α_2 approach to infinity. From eq. (8a), we can see that the finite element matrix equation now becomes

$$\delta \underset{\sim}{U}^T \left\{ \underset{\sim}{K}_2 + \omega^2 \underset{\sim}{M} \right\} \underset{\sim}{U} = 0 \tag{10}$$

where

$$\underset{\sim}{K}_2 = \underset{\sim}{K}_1 + \alpha_1 \underset{\sim}{a}(0) \underset{\sim}{a}^T(0) + \alpha_2 \underset{\sim}{a}'(0) \underset{\sim}{a}'^T(0) \tag{11}$$

The matrix $\underset{\sim}{K}$ in eq. (11) has been defined in eq. (5) and the superscript T denotes the transpose of a matrix (a vector). Since δu is arbitrary, $\delta \underset{\sim}{U}$ in eq. (10) is arbitrary, eq. (10) leads directly to the final matrix equation to be solved.

$$\left\{ \underset{\sim}{K}_2 + \omega^2 \underset{\sim}{M} \right\} \underset{\sim}{U} = 0 \tag{12}$$

It is then clear that the method of Lagrange multipliers, used in conjunction with the finite element method, will not only facilitate the solution formulations but also encompass a larger class of problems to be solved compared with the use of constrained variational statements. The applications of the same general concept can be extended further.

3. FROM UNCONSTRAINED VARIATIONS TO ADJOINT VARIATIONAL STATEMENTS.

We have noted that the variation δu in eq. (8) is quite independent of the function u itself and nothing will be changed if we simply replace δu with δv to emphasize this independence. This substitution, however, has suggested the adjoint variational principles. Let us consider

$$\delta J_3 = 0 \tag{13a}$$

$$J_3 = \int_0^\ell (E\,I\,u''v'' - Pu'v' + \omega^2 \rho A uv)\,dx$$

$$+ \alpha_1 u(0)v(0) + \alpha_2 u'(0)v(0) + \alpha_3 Pu'(\ell)v(\ell) \tag{13b}$$

Carrying out the variations, we have:

$$\delta J_3 = (\delta J_3)_u + (\delta J_3)_v \tag{14}$$

where

$$(\delta J_3)_u = \int_0^\ell (E\,I\,u''\delta v'' - Pu'\delta v' + \omega^2 \rho A u \delta v)\,dx$$

$$+ \alpha_1\, u(0)\delta v(0) + \alpha_2 u'(0)\delta v'(0) + \alpha_3 u'(\ell)\,\delta v(\ell) \tag{15a}$$

$$= \int_0^\ell (E\,I\,u'''' + Pu'' + \omega^2 \rho A u)\,\delta v\,dx$$

$$+ [E\,I\,u''\delta v' - (E\,I\,u''' + Pu' - \alpha_3 u')\,\delta v]_{x\,=\,\ell}$$

$$- \left[(E\ I\ u'' - \alpha_2 u')\ \delta v' - (E\ I\ u''' + Pu' + \alpha_1 u)\ \delta v \right]_{x\ =\ 0} \qquad (15b)$$

and

$$(\delta J_3)_v = \int_0^\ell (E\ I\ v''\ \delta u'' - Pv'\delta u' + \omega^2 \rho A v \delta u)\ dx$$

$$+ \alpha_1\ v(0)\ \delta u(0) + \alpha_2\ v'(0)\ \delta u'(0) + \alpha_3\ v(\ell)\ \delta u'(\ell) \qquad (16a)$$

$$= \int_0^\ell (E\ I\ v'''' + Pv'' + \omega^2 \rho A v)\ \delta u\ dx$$

$$+ \left[(E\ I\ v'' + \alpha_3 v)\ \delta u' - (E\ I\ v''' + P\ v')\ \delta u \right]_{x\ =\ \ell}$$

$$- \left[(E\ I\ v'' - \alpha_2\ v')\ \delta u' - (E\ I\ v''' + Pv' + \alpha_1\ v)\ \delta u \right]_{x\ =\ 0} \qquad (16b)$$

From eq. (15a), it is clear that a necessary and sufficient condition for $(\delta J_3)_u = 0$ is the problem defined by the following set of equations:

D.E. $$E\ I\ u'''' + Pu'' + \omega^2 \rho A u = 0 \qquad (17a)$$

B.C. $$E\ I\ u''(\ell) = 0 \qquad (17b)$$

$$E\ I\ u'''(\ell) + (P - \alpha_3)\ u'(\ell) = 0 \qquad (17c)$$

$$E\ I\ u''(0) - \alpha_2\ u'(0) = 0 \qquad (17d)$$

$$E\ I\ u'''(0) + P\ u'(0) + \alpha_1\ u(0) = 0 \qquad (17e)$$

Now eqs.(9) has become a special case of eqs.(17) when $\alpha_3 = 0$. In addition, the problem defined by $(\delta J_3)_v = 0$ of eqs. (16) is called the adjoint problem to eqs. (17). For $\alpha_3 = 0$, the adjoint problem is identical to the problem itself — hence, the self-adjoint system. Now, considering

$$\alpha_3 = k\ P \qquad (18)$$

in eq. (17c), we have

$$E\ I\ u'''(\ell) - K\ P\ u'(\ell) = 0 \qquad (19)$$

$$K = k - 1 \qquad (20)$$

Eq. (19) defines the boundary condition of a general non-conservative load. It is also clear from eq. (19) that K is a dimensionless design constant which defines the small angle between the direction of the applied load P and the tangent of the deflected column at the end. Since $(\delta J_3)_u = 0$ alone defines the boundary value problem of eq. (17) and vice versa, we need not at all to be concerned with the adjoint problem. Now it is a simple matter to modify the finite element matrix equation as

$$\left\{ \underset{\sim}{K}_3 + \omega^2 \underset{\sim}{M} \right\} \underset{\sim}{U} = 0 \tag{22}$$

where

$$\underset{\sim}{K}_3 = \underset{\sim}{K}_2 + \alpha_3 \, \underset{\sim}{a}'(\ell) \, \underset{\sim}{a}^T(\ell) \tag{23}$$

4. FINITE ELEMENTS FOR INITIAL AND INITIAL-BOUNDARY VALUE PROBLEMS.

(1) A Forced Vibration Problem. Let us first consider a problem of "one" degree of freedom, i.e., a mass-spring system. The differential equation and initial conditions are

$$m \ddot{u} + k u = f(t), \quad 0<t<T \tag{24a}$$

$$u(0) = u_0 \tag{24b}$$

$$\dot{u}(0) = u_1 \tag{24c}$$

where $u(t)$ is the displacement of the mass centre from its equilibrium position, m, the amount of mass and k, the spring constant. The function $f(t)$ is given, so are the constants a and b. The constant T appeared in the bounds of eq. (24a) is any given positive number other than infinity. In order to formulate approximate solutions for eqs. (24) the way we did in the previous section, let us consider a more general case

$$m \ddot{u} + k u = f(t) \tag{25a}$$

$$\dot{u}(T) - \alpha \left[u(0) - u_0 \right] = 0 \tag{25b}$$

$$\dot{u}(0) = u_1 \tag{25c}$$

where α is a parameter, obviously eqs. (25) reduce to (24) when α approaches to ∞. Now, with eqs. (25), we are able to write an unconstrained variational statement as follows:

$$\delta J_4 = 0 \tag{26a}$$

where

$$J_4 = \int_0^T [\, - m\dot{u}\dot{v} + kuv - f(t)v \,]\, dt$$

$$+ m\alpha\, [u(0) - u_0]\, v(T) - m u_1\, v(0) \qquad . \tag{26b}$$

Since

$$(\delta J_4)_u = \int_0^T [\, - m\dot{u}\delta\dot{v} + ku\delta v - f(t)\, \delta v \,]\, dt$$

$$+ m\alpha\, [\, u(0) - u_0 \,]\, \delta v(T) - m u_1 \delta v(0) \tag{27a}$$

$$= \int_0^T [\, m\ddot{u} + ku - f(t) \,]\, \delta v\, dt$$

$$- m \left\{ \dot{u}(T) - \alpha\, [u(0) - u_0] \right\} \delta v(T)$$

$$+ m\, [\dot{u}(0) - u_1 \,]\, \delta v(0) \tag{27b}$$

The already familiar form of eqs. (27) state that (a), $(\delta J)_u = 0$ is a necessary and sufficient condition for eqs. (25), and (b), eq. (27a) provides us the finite element matrix equation. Thus, if we assume as before that

$$u(t) = \underset{\sim}{a}^T(t)\, \underset{\sim}{U}$$

$$v\,(t) = \underset{\sim}{a}^T(t)\, \underset{\sim}{V}$$

Eq. (27a) yields

$$\delta \underset{\sim}{V}^T\, \underset{\sim}{K}_4\, \underset{\sim}{U} = \delta \underset{\sim}{V}^T\, \underset{\sim}{F} \tag{28}$$

where

$$\underset{\sim}{K}_4 = \int_0^T (\, -m\, \dot{\underset{\sim}{a}}\, \dot{\underset{\sim}{a}}^T + k\, \underset{\sim}{a}\, \underset{\sim}{a}^T)\, dt$$

$$+ m\alpha\, \underset{\sim}{a}(T)\, \underset{\sim}{a}^T(0) \tag{29}$$

and

$$\underset{\sim}{F} = \int_0^T f(t)\, \underset{\sim}{a}\, dt + m\alpha u_0\, \underset{\sim}{a}(t) + m\, \dot{u}_0\, \underset{\sim}{a}(0) \tag{30}$$

Again, since $\delta \underset{\sim}{V}$ is unconstrained eq. (28) leads directly to

$$\underset{\sim}{K}_4\, \underset{\sim}{U} = \underset{\sim}{F} \tag{31}$$

which is the final equation to be solved.

(2) A Heat Conduction Problem. The one dimensional transient heat conduct problem can be described by the equation

$$\frac{\partial}{\partial x} (K \frac{\partial u}{\partial x}) - \rho c \frac{\partial u}{\partial t} - f(x,t) = 0 \tag{32a}$$

Boundary and initial conditions are

$$u(0,t) = g_0(t) \tag{32b}$$

$$u(L,t) = g\ (t) \tag{32c}$$

$$u(x,0) = h\ (x) \tag{32d}$$

where

K = thermal conductivity

ρ = material density

c = specific heat

$f(x,t)$ = heat source function

and

$g_0(t)$, $g_1(t)$ and $h(x)$ are prescribed functions

Let us consider

$$\delta J_5 = 0 \tag{33a}$$

$$\begin{aligned} J_5 = - \int_0^L \int_0^T & [K \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \rho c \frac{\partial u}{\partial t} V + f(x,t)\ v]\ dt\ dx \\ & + \int_0 \alpha K\ [u(L,t) - g_1(t)]\ v(L,t)\ dt \\ & - \int_0^T \alpha K\ [u(0,t) - g_0(t)]\ v(0,t)\ dt \\ & - \int_0^L \rho c\ [u(x,0) - h(x)]\ v(x,0)\ dx \end{aligned} \tag{33b}$$

since

$$\begin{aligned} (\delta J_5)_u = \int_0^L \int_0^T & K \frac{\partial u}{\partial x}\ \delta (\frac{\partial u}{\partial x}) + \rho c \frac{\partial u}{\partial t}\ \delta v + f(x,t)\ \delta v]\ dx\ dt \\ & + \int_0^T \alpha\ K\ [U(L,t) - g_1(t)]\ \delta v(L,t)\ dt \\ & \int_0^T \alpha\ K\ [u(0,t) - g_0(t)]\ \delta v(0,t)\ dt \\ & \int_0^L \rho c\ [u(x,0) - h(x)]\ \delta v(x,0)\ dx \end{aligned} \tag{34a}$$

$$= \int_0^L\int_0^T [\frac{\partial}{\partial x}(K\frac{\partial u}{\partial x}) - \rho c\frac{\partial u}{\partial t} - f(x,t)]\delta v\, dxdt$$

$$- \int_0^T K\left\{\frac{\partial u(L,t)}{\partial x} - \alpha[u(0,t)-g_1(t)]\right\}\delta v(L,t)\, dt$$

$$+ \int_0^T K\left\{\frac{\partial u(0,t)}{\partial x} - \alpha[u(0,t) - g_0(t)]\right\}\delta v(0,t)\, dt$$

$$+ \int_0^L \rho c\,[u(x,0) - h(x)]\,\delta v(x,0)\, dx \qquad (34b)$$

it is clear that $(\delta J_5)_u = 0$ is a necessary and sufficient condition for eqs (32) as $\alpha \to \infty$ and eq. (34a) provides the finite element matrix equation. We can write from eq. (34a),

$$- \int_0^L\int_0^T [K\frac{\partial u}{\partial x}\,\delta(\frac{\partial v}{\partial x}) + \rho c\frac{\partial u}{\partial t}\,\delta v]\, dxdt$$

$$+ \int_0^T \alpha K\,[u(L,t)\,\delta v(L,t) - u(0,t)\,\delta v(0,t)]\, dt$$

$$+ \int_0^L \rho\, c\, u(x,0)\,\delta v(x,0)\, dt$$

$$= \int_0^L\int_0^T f(x,t)\,\delta v\, dxdt$$

$$+ \int_0^T \alpha K\,[g_1(t)\,\delta v(L,t)\; g_0(t)\,\delta v(0,t)]\, dt$$

$$+ \int_0^L \rho\, c\, h(x)\,\delta v(x,0)\, dx \qquad (35)$$

Now, let

$$u(x,t) = \underset{\sim}{a}^T(x,t)\,\underset{\sim}{U} \qquad (36a)$$

$$v(x,t) = \underset{\sim}{a}^T(x,t)\,\underset{\sim}{V} \qquad (36b)$$

in the usual manner, we have

$$\delta\underset{\sim}{V}^T\,\underset{\sim}{K}\,\underset{\sim}{U} = \delta\underset{\sim}{V}^T\,\underset{\sim}{F} \qquad (37)$$

$$\underset{\sim}{K} = - \int_0^L\int_0^T (K\,\underset{\sim}{a}_{,x}\,\underset{\sim}{a}^T_{,x} + \rho\, c\,\underset{\sim}{a}\,\underset{\sim}{a}^T_{,t})\, dx\, dt$$

$$+ \int_0^T \alpha K\,[\underset{\sim}{a}(L,t)\,\underset{\sim}{a}(L,t) - \underset{\sim}{a}(0,t)\,\underset{\sim}{a}^T(0,t)]\, dt$$

$$+ \int_0^L \rho\, c\,\underset{\sim}{a}(x,0)\,\underset{\sim}{a}^T(x,0)\, dx \qquad (38)$$

and

$$F = \int_0^L\int_0^T f(x,t)\ a(x,t)\ dxdt$$
$$+ \int_0^T \alpha K\ [g_1(t)\ a(L,t) - g_0(t)\ a(0,t)]\ dt$$
$$+ \int_0^L \rho\ c\ h(x)\ \underset{\sim}{a}(x,0)\ dx \tag{39}$$

Again, since $\delta \underset{\sim}{V}$ in eq. (37) is completely arbitrary, we arrive at the final matrix equation to be solved.

$$\underset{\sim}{K}\ \underset{\sim}{U} = \underset{\sim}{F} \tag{40}$$

(3) A Wave Propagation Problem. For a quite general wave propagation problem, the following system can be written.

$$\frac{\partial^2 u}{\partial x^2} - c^2 \frac{\partial^2 u}{dt^2} = f(x,t) \tag{41a}$$

$$u(0,t) = g_0(t) \tag{41b}$$

$$u(L,t) = g_1(t) \tag{41c}$$

$$u(x,0) = h_0(x) \tag{41d}$$

$$\dot{u}(x,0) = h_1(x) \tag{41e}$$

The extension of the previous formulation to this problem is straight forward. Let us consider

$$\delta J_6 = 0 \tag{42a}$$

where

$$J_6 = \int_0^L\int_0^T [-\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + c^2 \frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\ f(x,t)\ v]\ dxdt$$
$$- \alpha \int_0^T [u(L,t) - g_1(t)]\ v(L,t)\ dt$$
$$+ \alpha \int_0^T [u(0,t) - g_0(t)]\ v(0,t)\ dt$$
$$- \alpha \int_0^L [u(x,0) - h_0(x)]\ v(x,T)\ dx$$
$$+ \int_0^L [u(x,0) - h_1(x)]\ v(x,0)\ dx \tag{42b}$$

Again,

$$(\delta J_6)_u = \int_0^L\int_0^T [- \frac{\partial u}{\partial v} \delta(\frac{\partial v}{\partial x}) + c^2 \frac{\partial u}{\partial t} \delta (\frac{\partial v}{\partial t}) - f(x,t)\delta v] \, dxdt$$

$$- \alpha \int_0^T [u(L,t) - g_1(t)] \ \delta v(L,t) \, dt$$

$$+ \alpha \int_0^T [u(0,t) - g_0(t)] \ \delta v(0,t) \, dt$$

$$- \alpha \int_0^L [u(x,0) - h_0(x)] \ \delta v(x,T) \, dx$$

$$+ \int_0^L [u(x,0) - h_1(x)] \ \delta v(x,0) \, dx \qquad (43a)$$

$$= \int_0^L\int_0^T [\frac{\partial^2 u}{\partial x^2} - c^2 \frac{\partial^2 u}{\partial t^2} - f(x,t)] \ v \, dxdt$$

$$+ \int_t \left\{ \frac{\partial u}{\partial x}(L,t) - \alpha [u(L,t) - g_1(t)] \right\} \delta v(L,t) \, dt$$

$$- \int_t \left\{ \frac{\partial u}{\partial x}(0,t) - \alpha [u(0,t) - g_0(t)] \right\} \delta v(0,t) \, dt$$

$$+ \int_V \left\{ \frac{\partial u}{\partial t}(x,t) - \alpha [u(x,0) - h_0(x)] \right\} \delta v(x,T) \, dx$$

$$- \int_V [\frac{\partial u}{\partial t}(x,0) - h_1(x)] \ \delta v(x,0) \, dx \qquad (43b)$$

From eqs. (43), it is again clear that $(\delta J_6)_u = 0$ is a necessary and sufficient condition for eqs. (41) as $\alpha \to \infty$ and that eq. (43a) will yield the finite element matrix equation. From (43a) one has:

$$\int_0^L\int_0^T \left\{ - \frac{\partial u}{\partial x} \delta(\frac{\partial v}{\partial x}) + c^2 \frac{\partial u}{\partial t} \delta (\frac{\partial v}{\partial t}) \right\} dx \, dt$$

$$- \alpha \int_t u(L,t) \ \delta v(L,t) \, dt + \alpha \int_t u(o,t) \ \delta v(0,t) \, dt$$

$$- \alpha \int_V u(x,0) \ \delta v(x,T) \, dx$$

$$= \iint f(x,t) \ \delta v(x,t) \, dx \, dt$$

$$\alpha \int_t g_1(t)\ \delta v(L,t)\ dt + \alpha \int_t g_0(t)\ \delta v(0,t)\ dt$$

$$- \alpha \int_V h_0(x)\ \delta v(x,T)\ dx + \int_x h_1(x)\ \delta v(x,0)\ dx \tag{44}$$

Again, let

$$u(x,t) = \underset{\sim}{a}^T(x,t)\ \underset{\sim}{U} \tag{45a}$$

$$v(x,t) = \underset{\sim}{a}^T(x,t)\ \underset{\sim}{V} \tag{45b}$$

Eq. (44) becomes, in matrix form,

$$\delta\ \underset{\sim}{V}^T\ \underset{\sim}{K}\ \underset{\sim}{U} = \delta\ \underset{\sim}{V}^T\ \underset{\sim}{F} \tag{46}$$

where

$$\underset{\sim}{K} = \int_0^T \int_0^L (-\ \underset{\sim}{a}'\ \underset{\sim}{a}'^T + c^2\ \dot{\underset{\sim}{a}}\ \dot{\underset{\sim}{a}}^T)\ dxdt$$

$$- \alpha \int_0^T \underset{\sim}{a}(L,t)\ \underset{\sim}{a}^T(L,t)\ dt + \alpha \int_0^T \underset{\sim}{a}(0,t)\ \underset{\sim}{a}^T(0,t)\ dt$$

$$- \alpha \int_0^L \underset{\sim}{a}(x,t)\ \underset{\sim}{a}^T(x,0)\ dt \tag{47}$$

$$\underset{\sim}{F} = \int_0^T \int_0^L f(x,t)\ \underset{\sim}{a}(x,t)\ dxdt$$

$$- \alpha \int_0^T g_1(t)\ \underset{\sim}{a}(L,t)\ dt + \alpha \int_0^T g_0(t)\ \underset{\sim}{a}(0,t)\ dt$$

$$+ \alpha \int_0^L h_0(x)\ \underset{\sim}{a}(x,T)\ dt + \int_0^L h_1(x)\ \underset{\sim}{a}(x,0)\ dt \tag{48}$$

Due to the arbitrariness of $\delta \underset{\sim}{V}$, eq. (46) leads directly to the final matrix equation

$$\underset{\sim}{K}\ \underset{\sim}{U} = \underset{\sim}{F} \tag{49}$$

5. NUMERICAL DEMONSTRATIONS. Several numerical examples will be given in this section to demonstrate the application of the formulation described so far.

(1) Forced Vibration. We shall consider a special case of the forced vibration problem formulated earlier. The forcing function in eqs. (24) is taken to be a cosine function thus, rewrite eqs. (24),

$$m\ \ddot{u} + k\ u = f_0 \cos \omega_f t \tag{50a}$$

$$u(0) = u_0 \tag{50b}$$

$$\dot{u}(0) = u_1 \tag{50c}$$

where u_0, u_1, f_0 and ω_f are given constants. In the finite element formulation, we shall replace eqs. (50) with the following set

$$m\, \ddot{u} + k\, u = f_0 \cos \omega_f t \tag{51a}$$

$$\dot{u}(t) - \alpha\, [u(0) - u_0] = 0 \tag{51b}$$

$$\dot{u}(0) - u_1 = 0$$

thus, eqs. (50) becomes a special case of (51) as $\alpha \to \infty$. It is convenient to nondimensionalize the independent variable t and let

$$\tau = t/T \tag{52}$$

In terms of τ, eqs. (51) become

$$\ddot{u} + T^2\, \omega^2\, u = f_1 \cos\, (T\, \omega_f\, \tau) \tag{53a}$$

$$\dot{u}(1) - T\, \alpha\, [u(0) - u_0] = 0 \tag{53b}$$

$$\dot{u}(0) - T\, u_1 = 0 \tag{53c}$$

where

$$f_1 = T^2\, f_0/m \quad \omega^2 = k/m \tag{54}$$

The exact solution for eqs. (53) can be easily written as

$$u(\tau) = A \cos\, (T\, \omega \cdot \tau) + B \sin\, (T\, \omega \cdot \tau) + \eta \cos\, (T\, \omega_f \cdot \tau) \tag{55}$$

with

$$\eta = \frac{f_0}{m(\omega^2 - \omega_f^2)}\,, \quad \beta = \frac{u_1}{\omega}$$

$$A = \frac{\alpha\, u_0 + T\, u_1 \cos\, (T\omega) - \eta\, [\alpha + T\omega_f \sin\, (T\omega_f)]}{\alpha + T\, \omega \sin\, (T\, \omega)} \tag{56}$$

To solve eqs. (53) using finite elements, one begins with the variational statement:

$$\delta\, J = 0 \tag{57a}$$

$$J = \int_0^1 [-\, \dot{u}\, v + T^2\, \omega^2 uv - f(\tau)\, v]\, d\tau$$

$$+ T\alpha\,[u(0) - u_0]\; v(1) - T\, u_1\, v(0) \tag{57b}$$

Now that

$$(\delta J)_u = 0 \tag{58a}$$

$$= \int_0^1 [-\dot{u}\,\delta\,\dot{v} + T^2\,\omega^2\,u\delta v - f(\tau)\delta\, v]$$

$$+ T\,\alpha\,[u(0) - u_0]\; v(1) - Tu_1\delta v(0) \tag{58b}$$

$$= \int_0^1 [\ddot{u} + T^2\,\omega^2\,u - f(\tau)]\;\delta v\; dt$$

$$- \{u(0) - \alpha\, T[u(0) - u_0]\}\;\delta v(1)$$

$$+ \{\,u(0) - T\,u_1\}\;\delta v(0) \tag{58c}$$

From eq. (58b), one has

$$\int_0^1 [-\dot{u}\,\delta\,\dot{v} + T^2\omega^2\,u\,\delta\,v]\;dt + \alpha T\,u(0)\;\delta v(1)$$

$$= \int_0^1 f(\tau)\;\delta v\;d\tau + \alpha\,T\,u_0\;\delta v(1) + T\,u_1\;\delta v(0) \tag{59}$$

with

$$u(\tau) = \underset{\sim}{a}^T(\tau)\;\underset{\sim}{U}$$

$$v(\tau) = \underset{\sim}{a}^T(\tau)\;\underset{\sim}{V} \tag{60}$$

eq. (59) leads to

$$\delta\,\underset{\sim}{V}^T\,\underset{\sim}{K}\,\underset{\sim}{U} = \delta\,\underset{\sim}{V}^T\,\underset{\sim}{F}$$

or

$$\underset{\sim}{K}\,\underset{\sim}{U} = \underset{\sim}{F} \tag{61}$$

where

$$K = \int (-\,\dot{\underset{\sim}{a}}\,\dot{\underset{\sim}{a}}^T + T^2\,\omega^2\,\underset{\sim}{a}\,\underset{\sim}{a}^T)\;dt$$

$$+ \alpha\,T\,\underset{\sim}{a}(1)\;\underset{\sim}{a}(0) \tag{62}$$

$$\underset{\sim}{F} = \int_0^1 f(\tau)\;\underset{\sim}{a}\;d\tau + \alpha T u_0\,\underset{\sim}{a}(1) + T\,u_1\;\underset{\sim}{a}(0) \tag{63}$$

The results obtained from this finite element formulation are compared with the exact solutions as shown in Tables 1 - 3. The values of the parameters chosen for these data are $k = 1.0$, $m = 1.0$, $f_0 = 1.0$, $\omega_f = 0.5$, $u_0 = 1.0$, $\dot{u}_0 = 1.0$ the number of elements used is ten. The calculated u and $\dot{u}$ for $T = 2.0$, 10.0 and 20.0 are given in Table 1, 2, 3 and 4 respectively. The forcing function $\cos\omega_f t$ and the solution $u(t)$ are also plotted in the range $0 \lesssim t \lesssim 20$ as shown in Figure 1.

(2) Solutions to a Transient Heat Conduction Problem. As another numerical example, we shall take the nondimensional heat transfer problem defined by the following set:

TABLE 1. Solutions to the Forced Vibration Problem Using FE-UVF Compared with Exact Solutions (in Parentheses) $0 \leq t \leq 2.0$

t	u(t)	Exact	$\dot{u}(t)$	Exact
0	1.000 000 0	(1.000 000 0)	1.000 00	(1.000 00)
0.2	1.198 652 6	(1.198 652 7)	0.979 74	(0.979 73)
0.4	1.389 153 7	(1.389 153 4)	0.918 43	(0.918 42)
0.6	1.563 313 2	(1.563 312 6)	0.816 55	(0.816 54)
0.8	1.713 202 9	(1.713 201 8)	0.676 22	(0.676 21)
1.0	1.831 481 7	(1.831 480 3)	0.501 18	(0.501 18)
1.2	1.911 702 4	(1.911 700 6)	0.296 62	(0.296 61)
1.4	1.948 585 6	(1.948 583 6)	0.068 98	(0.068 97)
1.6	1.938 251 2	(1.938 249 1)	- 0.174 24	(-0.174 25)
1.8	1.878 396 9	(1.878 395 0)	- 0.424 82	(-0.424 80)
2.0	1.768 416 1	(1.768 416 1)	- 0.674 13	(-0.674 03)

TABLE 2. Solutions to the Forced Vibration Problem Using FE-UVF Compared with Exact Solutions (in Parentheses) $0 \leq t \leq 10.0$

t	u(t)		$\dot{u}(t)$	
0	1.000	(1.000)	1.004	(1.000)
1.0	1.832	(1.831)	0.505	(0.501)
2.0	1.770	(1.768)	- 0.675	(-0.674)
3.0	0.566	(0.565	- 1.614	(-1.608)
4.0	- 1.094	(-1.094)	- 1.518	(-1.512)
5.0	- 2.123	(-2.122)	- 0.435	(-0.435)
6.0	- 1.920	(-1.919)	0.778	(0.773)
7.0	- 0.843	(-0.843)	1.213	(1.207)
8.0	0.167	(0.166)	0.690	(0.689)
9.0	0.436	(0.435)	- 0.126	(-0.122)
10.0	0.114	(0.114)	- 0.385	(-0.381)

TABLE 3. Solution to the Forced Vibration Problem Using FE-UVF Compared with Exact Solutions (in Parentheses) $0 \leq t \leq 20.0$

t	u(t)		$\dot{u}(t)$	
0	1.000	(1.000)	1.05	(1.00)
2.0	1.778	(1.768)	- 0.68	(-0.67)
4.0	- 1.097	(-1.094)	- 1.57	(-1.51)
6.0	- 1.928	(-1.919)	0.82	(0.77)
8.0	0.173	(0.166)	0.71	(0.69)
10.0	0.116	(0.114)	- 0.44	(-0.38)
12.0	0.453	(0.462)	0.88	(0.85)
14.0	1.956	(1.950)	0.06	(0.03)
16.0	- 0.156	(-0.162)	- 1.76	(-1.71)
18.0	- 2.199	(-2.186)	0.15	(0.14)
20.0	- 0.348	(-0.342)	1.10	(1.08)

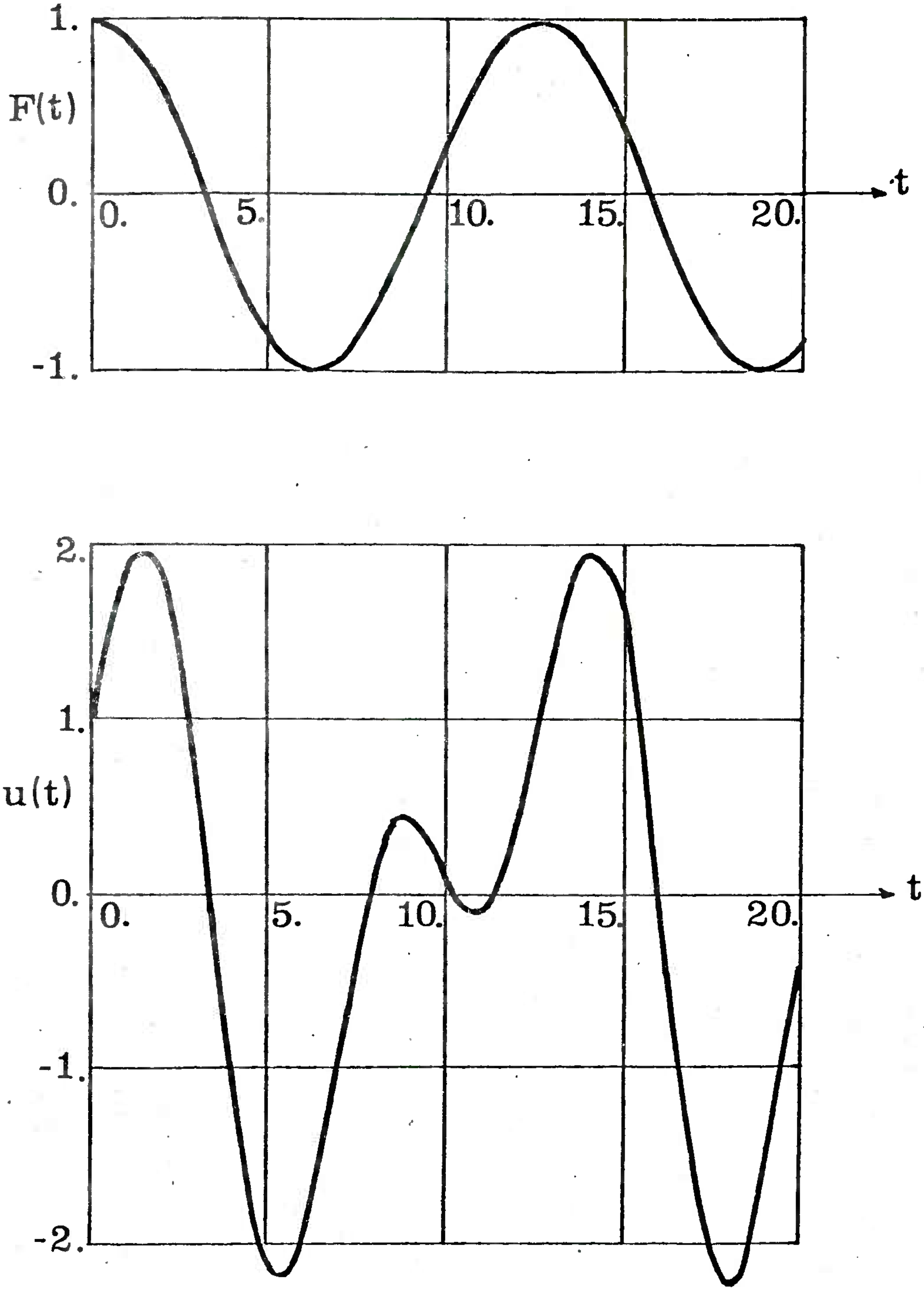

Figure 1. Forcing Function F(t) and Solution u(t) for the Vibration Problem

$$\text{D.E.:} \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \quad , \quad 0<x<1 \ ; \quad 0<t<T \tag{64a}$$

$$\text{B.C.:} \quad u(0),t) = 1, \ \frac{\partial u}{\partial x} \ (1,t) = 0 \tag{64b,c}$$

$$\text{I.C.:} \quad u(x,0) = 0 \tag{64d}$$

where T is any given finite real positive number. To facilitate computation, it is desirable to change the independent variable t into τ such that

$$\tau = t/T \tag{65}$$

thus, the system of eqs. (64) becomes

$$\text{D.E.:} \quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{T}\frac{\partial u}{\partial \tau} = 0 \ , \ 0<x<1 \ ; \ 0<\tau<1 \tag{66a}$$

$$\text{B.C.:} \quad u(0,\tau) = 1 \ ; \ \frac{\partial u}{\partial x} (1,\tau) = 0 \tag{66b,c}$$

$$\text{I.C.:} \quad u(x,0) = 0 \tag{66d}$$

According to our unconstrained variational formulation, this system is again replaced by the following:

$$\text{D.E.:} \quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{T}\frac{\partial u}{\partial \tau} = 0 \ , \ 0<x<1 \ ; \ 0<\tau<1 \tag{67a}$$

$$\text{B.C.:} \quad \frac{\partial u}{\partial x} (0,\tau) + \alpha \ [u(0,\tau) - 1] = 0 \tag{67b}$$

$$\frac{\partial u}{\partial x} \ (1,\tau) = 0 \tag{67c}$$

$$\text{I.C.:} \quad u(x,0) = 0 \tag{67d}$$

Clearly, eqs. (67) reduces to (66) as $\alpha \to \infty$. The variational statement can be written as

$$\delta J = 0 \tag{68a}$$

where

$$J = - \int_0^1\int_0^1 \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{T}\frac{\partial u}{\partial \tau} v\right) dxdt + \alpha\int_0^1 [u(0,\tau) - 1] \ v(0,\tau) \ d\tau + \int_0^1 u(x,0) \ v(x,0) \ dx \tag{68b}$$

Due to the fact that $v(x,\tau)$ is unconstrained, it is a simple matter to show that

$$(\delta J)_u = 0 \tag{69}$$

is a necessary and sufficient condition of eqs. (67). Now the finite element matrix equations can be obtained from eq. (69).

$$(\delta J)_u = - \int_0^1 \int_0^1 [\frac{\partial u}{\partial x} \delta(\frac{\partial v}{\partial x}) + \frac{1}{T} \frac{\partial u}{\partial \tau} \delta v] \, dxdt$$

$$+ \int_0^1 [u(0,\tau) - 1] \, \delta v(0,\tau) \, dt$$

$$+ \int_0^1 u(x,0) \, \delta v(x,0) \, dx = 0 \tag{70}$$

or,

$$- \int_0^1 \int_0^1 [\frac{\partial u}{\partial x} \delta(\frac{\partial v}{\partial x}) + \frac{1}{} \frac{\partial u}{\partial \tau} \delta v] \, dxdt$$

$$+ \alpha \int_0^1 u(0,\tau) \, \delta v(0,\tau) \, dt + \int_0^1 u(x,0) \, \delta v(x,0) \, dx$$

$$= \alpha \int_0^1 \delta v(0,\tau) \, d\tau \tag{71}$$

Using the usual procedure of discretization and the assumption of displacement functions, the final finite element matrix equation evidently can be derived from eq. (71). We shall omit the details here. The computational results are presented in Tables 4 and 5. The finite element grid scheme used is shown in Figure 2. As clearly shown in those tables, excellent agreement exists between the FE-UVF approach and the series solution. It is noted that the approximate solutions are less accurate invariably as they approach the initial time $t = 0$. This is probably due to the discontinuity of the initial boundary data at $x = 0$, $t = 0$.

TABLE 4. Transient Heat Transfer Solutions u(x,t) Using FE-UVF Compared with Exact Series Solutions (in Parentheses)

$0 < t < T = 1.00$

t \ x	0	0.2	0.4	0.6	0.8	1.0
0.2	1.000 (1.000)	0.754 (0.757)	0.583 (0.496)	0.370 (0.405)	0.264 (0.284)	0.228 (0.179)
0.4	1.000 (1.000)	0.855 (0.853)	0.713 (0.721)	0.622 (0.616)	0.552 (0.549)	0.516 (0.526
0.6	1.000 (1.000)	0.910 (0.910)	0.828 (0.830)	0.767 (0.767)	0.725 (0.724)	0.708 (0.710)
0.8	1.000 (1.000)	0.945 (0.945)	0.896 (0.896)	0.857 (0.857)	0.832 (0.832)	0.823 (0.823)
1.0	1.000 (1.000)	0.967 (0.967)	0.937 (0.937)	0.913 (0.913)	0.897 (0.897)	0.892 (0.892)

TABLE 5. Transient Heat Transfer Solutions u(x,t) Using FE-UVF Compared with Exact Series Solutions (in Parentheses)

$0 < t < T = 0.05$

t \ x	0	0.2	0.4	0.6	0.8	1.0
0.01	1.000 (1.000)	0.144 (0.157)	0.014 (0.005)	0.002 (0.000)	0.000 (0.000)	0.000 (0.000)
0.02	1.000 (1.000)	0.315 (0.317)	0.047 (0.046)	(0.003) (0.003)	(0.000) (0.000)	(0.000 (0.000)
0.03	1.000 (1.000)	0.413 (0.414)	0.103 (0.102)	0.015 (0.014)	0.001 (0.001)	0.000 (0.000)
0.04	1.000 (1.000)	0.479 (0.480)	0.157 (0.157)	0.034 (0.034)	0.005 (0.005)	0.001 (0.001)
0.05	1.000 (1.000)	0.527 (0.527)	0.206 (0.206)	0.058 (0.058)	0.012 (0.012)	0.003 (0.003)

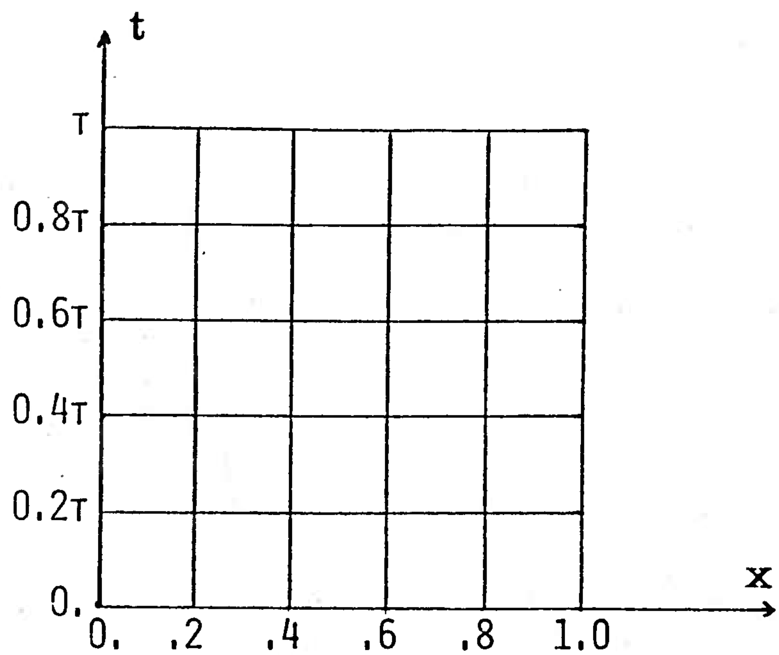

Figure 2. Finite Element Grid Scheme Used for a Transient Heat Conduction Problem

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